stichting mathematisch centrum

MC

AFDELING TOEGEPASTE WISKUNDE TN 102/82 JUNI
(DEPARTMENT OF APPLIED MATHEMATICS)

B. HARSOYO & N.M. TEMME

A FUNCTIONAL EQUATION FOR A SERIES RELATED TO THETA FUNCTIONS

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 44A15, 40A25, 39B05

A functional equation for a series related to theta functions

by

B. Harsoyo *) & N.M. Temme

ABSTRACT

In this report we consider the series

$$f(x) = -\sum_{n=1}^{\infty} \log(1-e^{-nx})$$

for positive values of x. By using Mellin transform techniques the series is written as an integral, from which an interesting functional relation of f(x) is derived. The result is

$$f(x) = \frac{\pi^2}{6x} + \tfrac{1}{2}\log\frac{x}{2\pi} - \frac{x}{24} + f(4\pi^2/x),$$

which holds true when x is replaced by $z = x + iy$, $x > 0$. This equation enables computation for small values of x, since in that event $f(4\pi^2/x)$ is exponentially small.

KEY WORDS & PHRASES: *Mellin transform, transformation of series, functional equation*

*) Permanent address: Universitas Indonesia, Jakarta

1. INTRODUCTION

At the Dutch Mathematical Congress of April, 1982, Drs. B.R. Damsté of the Landbouwhogeschool in Wageningen discussed a numerical method (based on a Van Wijngaarden transformation) for computing slowly convergent series. As an example for applying the method he mentioned the series

$$(1.1)\qquad S(x) = \sum_{n=1}^{\infty} \log \coth nx, \qquad x > 0,$$

which arose at his institute in hydrological problems.

In this paper we consider the above series from an analytical point of view. As remarked by Damsté, for small values of x the series (1.1) converges slowly and he found that in that event the following approximation can be used

$$(1.2)\qquad S(x) = \pi^2/(8x) + \tfrac{1}{2} \log (x/2\pi) + R(x),$$

where R(x) is rather small. He obtained the first two terms in this expansion by comparing (1.1) with an earlier investigated series of which the two terms represented the exact sum. We will show why (1.2) gives a good approximation for small values of x. That is, we will prove that R(x) of (1.2) is exponentially small compared with the first two terms. As follows from (1.7), its order is $\mathcal{O}(\exp(-\pi^2/x))$, which makes plausible that (1.2) gives more than about nine significant digits for $x \in (0,\tfrac{1}{2})$ when R(x) is neglected.

To obtain the results we consider the related series

$$(1.3)\qquad f(x) = -\sum_{n=1}^{\infty} \log(1-e^{-nx}), \qquad x > 0,$$

which appears to be a basic function for the properties of S(x). By writing

$$(1.4)\qquad \log \coth nx = \log \frac{1+e^{-2nx}}{1-e^{-2nx}} = \log \frac{1-e^{-4nx}}{(1-e^{-2nx})^2}$$

it follows that

$$(1.5) \qquad S(x) = 2f(2x) - f(4x).$$

The main result of the paper is a functional equation for f(x). We will prove that f(x) satisfies

$$(1.6) \qquad f(x) = \frac{\pi^2}{6x} + \tfrac{1}{2}\log\frac{x}{2\pi} - \frac{x}{24} + f(4\pi^2/x).$$

To obtain this result we transform (1.3) into an integral by using Mellin transform techniques. The first three terms of the right-hand side of (1.6) are residues of a Barnes-type contour integral.

When the result (1.6) is established it follows by using (1.5) that S(x) satisfies

$$S(x) = 2[\frac{\pi^2}{12x} + \tfrac{1}{2}\log\frac{x}{\pi} - \frac{x}{12} + f(2\pi^2/x)] - [\frac{\pi^2}{24x} + \tfrac{1}{2}\log\frac{2x}{\pi} - \frac{x}{6} + f(\pi^2/x)],$$

giving (1.2) with

$$(1.7) \qquad R(x) = 2f(2\pi^2/x) - f(\pi^2/x).$$

The title of the paper indicates a connection with theta functions. In fact we have with $q = \exp(-x)$ the representations

$$f(x) = -\log\prod_{n=1}^{\infty}(1-q^n), \qquad S(x) = \log\prod_{n=1}^{\infty}\frac{1+q^{2n}}{1-q^{2n}}.$$

The infinite products appear indeed in some formula's for theta functions. See, for instance, WHITTAKER & WATSON (1927, section 21.3). This relation will not be exploited, however.

In section 2 we give a formula for f(x) which is more convenient for computations of f(x), especially for large values of x. In section 3 we derive the integral representation for f(x), and in section 4 we prove equation (1.6). This formula is very useful for the computation of f(x) for small values of x, since the term $f(4\pi^2/x)$ is exponentially small in that case.

2. A SERIES FREE OF LOGARITHMS

The logarithms in the terms of the series (1.1) and (1.3) make these representations less attractive for computations, especially when x is large. Therefore we transform the series into series which are free of logarithms and of which the convergence is as fast as the original series.

Using the well-known series

$$\log(1-z) = -\sum_{m=1}^{\infty} \frac{z^m}{m}$$

we have

$$\sum_{n=1}^{\infty} \log(1-e^{-nx}) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{e^{-nmx}}{m}$$

$$= -\sum_{m=1}^{\infty} \frac{1}{m}\frac{e^{-mx}}{1-e^{-mx}} = -\sum_{m=1}^{\infty} \frac{1}{m(e^{mx}-1)} .$$

This gives the representation

$$f(x) = \sum_{m=1}^{\infty} \frac{1}{m(e^{mx}-1)}, \qquad x > 0, \tag{2.1}$$

and for S(x) we obtain, using

$$\log\frac{1+z}{1-z} = 2\sum_{m=0}^{\infty} \frac{z^{2m+1}}{2m+1},$$

the expansion

$$S(x) = 2\sum_{m=0}^{\infty} \frac{1}{(2m+1)[\exp(2m+1)x-1]}, \qquad x > 0. \tag{2.2}$$

The inversion of the order of summation of the n- and m-series is permitted, since the double series converges absolutely when $x > 0$.

The series in (2.1) and (2.2) give a convenient representation for large values of x. The evaluation of the logarithmic functions in (1.1) and (1.3) may result in a large relative error in that case. For small values of x, (2.1) and (2.2) are just as useless as (1.1) and (1.3). However, (1.6) gives an interesting reflection formula for small x.

3. INTEGRAL REPRESENTATION OF S(x)

We use Mellin transform techniques to obtain an integral representation for f(x). Let us recall the definition of a Mellin transform for a function $f\colon (0,\infty) \to \mathbb{C}$, which is locally integrable in $(0,\infty)$. Then the Mellin transform of f evaluated at s is

$$(3.1) \qquad M[f,s] = \int_0^\infty f(t)t^{s-1}dt,$$

whenever the integral converges. Usually, the domain of definition of M[f,s] is a strip in the s-plane

$$(3.2) \qquad \alpha < \mathrm{Re}\, s < \beta$$

where α and β are real constants ($\alpha<\beta$, infinite values not excluded). In this strip M[f,s] is analytic. M[f,s] also denotes the function which is an analytic continuation of the function defined by (3.1) in the complex s-plane.

The inversion formula for Mellin transforms reads

$$(3.3) \qquad f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s}M[f,s]ds,$$

where c satisfies $\alpha < c < \beta$. Equation (3.3) is valid at all points $t \geq 0$ where f(t) is continuous. The Barnes-type contour in (3.3) (which is a vertical line in the strip (3.2)) can be deformed within the domain of holomorphy of M[f,s] and moved across poles, when we take into account the residues at these poles.

More information on Mellin transforms can be found in, for instance, SNEDDON (1972).

To apply the Mellin transform, we consider the series (2.1) and we compute

$$(3.4)\qquad M[\phi_m,s] = \int_0^\infty x^{s-1}\phi_m(x)dx, \qquad m = 1,2,3,\ldots,$$

where

$$(3.5)\qquad \phi_m(x) = \frac{1}{m(e^{mx}-1)}, \qquad m = 1,2,3,\ldots .$$

Since

$$\phi_m(x) \sim \frac{1}{m^2 x}, \qquad x \to 0^+$$

$$\phi_m(x) \sim \frac{1}{m} e^{-mx}, \qquad x \to +\infty$$

We conclude that (3.4) is defined in the half-plane Re $s > 1$. That is, the constants α and β of (3.2) satisfy $\alpha = 1$, $\beta = +\infty$, which values can be used for all $m \geq 1$.

The integral in (3.4) can be expressed in terms of the Riemann zeta function, which is defined by the equation

$$(3.6)\qquad \zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \qquad \text{Re } z > 1.$$

For other values of z it is defined by analytic continuation. It is holomorphic in the complex plane, except at the point $z = 1$, where it has a simple pole of residue 1. Substituting Euler's integral for the Gamma function in the form

$$p^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty e^{-pt} t^{z-1} dt, \qquad \text{Re } p > 0,$$

we obtain for (3.6) the integral

$$(3.7)\qquad \zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt \qquad (\text{Re } z > 1).$$

The inversion of the order of summation and integration is permitted when Re $z > 1$ (see OLVER (1974, p.54)).

It follows that (3.4) can be written as

$$M[\phi_m, s] = \frac{1}{m^{s+1}} \zeta(s)\ \Gamma(z)$$

and that the transform of (2.1) is

$$M[f,s] = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \zeta(s)\ \Gamma(s) = \zeta(s+1)\ \zeta(s)\ \Gamma(s).$$

Again, the inversion of the order of summation and integration is permitted for Re $s > 1$.

Now we apply the formula for the inversion of the Mellin transform (3.3) and the result is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s}\ \zeta(s+1)\ \zeta(s)\ \Gamma(s)\ ds \tag{3.9}$$

where c is any real number larger than 1.

This formula gives the desired integral representation for $f(x)$.

4. THE FUNCTIONAL EQUATION FOR f(x)

The integral (3.9) will be used to derive the functional equation (1.6). The contour of integration is moved to the left and then the residues of the meromorphic integrand are computed. A basic function here is

$$E(s) = (2\pi)^{-s}\ \Gamma(s)\ \zeta(s)\ \zeta(1+s) \tag{4.1}$$

and first we prove an interesting property of this function.

LEMMA 4.1. *The function* $E(s)$ *defined in* (4.1) *is even in* s.

PROOF. We use the reflection formulas for the zeta function:

$$\zeta(1-z) = 2^{1-z}\pi^{-z} \cos(\tfrac{1}{2}\pi z)\ \Gamma(z)\ \zeta(z),$$

which is proved in OLVER (1974, p.63). It follows that E(s) can be written as

$$(4.2) \qquad E(s) = (2\pi)^{-s}\, \zeta(1+s)\, \zeta(1-s)\, 2^{-1+s}\pi^{s}/\cos \tfrac{1}{2}\pi s = \frac{\zeta(1+s)\zeta(1-s)}{2\cos \frac{1}{2}\pi s}$$

from which the symmetry follows. □

The function E(s) is holomorphic in the half-plane Re s > 1. Due to the symmetry it is also holomorphic in the half-plane Re s < −1. In fact, the poles of Γ(s) are cancelled by the zeros of ζ(s) ζ(s+1) at s = −2,−3,−4,... . Recall that ζ(z) has a simple pole at z = 1. Hence E(s) (see (4.2)) has a double pole at s = 0. It also follows from (4.2) that E(s) has a simple pole at s = ±1.

Consequently, when we move the contour of integration in the integral

$$(4.3) \qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (2\pi/x)^{s}\, E(s)\, ds, \qquad c > 1,$$

(cf. (3.9) and (4.1)) we have to take into account the residues of the poles at s = ±1, s = 0. At the remaining points in the complex s-plane the integrand is analytic.

Now we compute the residues of the poles.

(i) s = 1; the residue is (we use (4.1) and the fact that the residue of ζ(s) at s = 1 is 1) $\lim_{s\to 1} (\frac{2\pi}{x})\, E(s)(s-1) = \frac{1}{x}\,\zeta(2)$. By substituting z = 2, the series in (3.6) reduces to the well-known series

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

Hence the residue of $(2\pi/x)^s\, E(s)$ is $\pi^2/(6x)$.

(ii) s = −1; $(s^2-1)\, E(s)$ is even. Therefore, using the previous result,

$$\lim_{s\to -1} (s+1)(\tfrac{2\pi}{x})^{s}E(s) = \lim_{s\to -1} (s^2-1)E(s)(\tfrac{2\pi}{x})^{s}\frac{1}{s-1} =$$

$$\lim_{s\to +1} (s^2-1)E(s)(\tfrac{2\pi}{x})^{-s}\,\frac{-1}{s+1} =$$

$$- \lim_{s\to 1} (s-1)E(s)\left(\frac{2\pi}{x}\right)^s \left(\frac{2\pi}{x}\right)^{-2s} = - \frac{\pi^2}{6x} \frac{x^2}{4\pi^2} = - \frac{x}{24} .$$

(iii) $s = 0$; since

$$\zeta(s) = \frac{1}{s-1} + O(1), \qquad s \to 1,$$

and since $E(s)$ is even we have (see (4.2))

$$E(s) = - \frac{1/2}{s^2} [1+O(s^2)], \qquad s \to 0.$$

Hence

$$(2\pi/x)^s E(s) = e^{s \log \frac{2\pi}{x}} E(s)$$

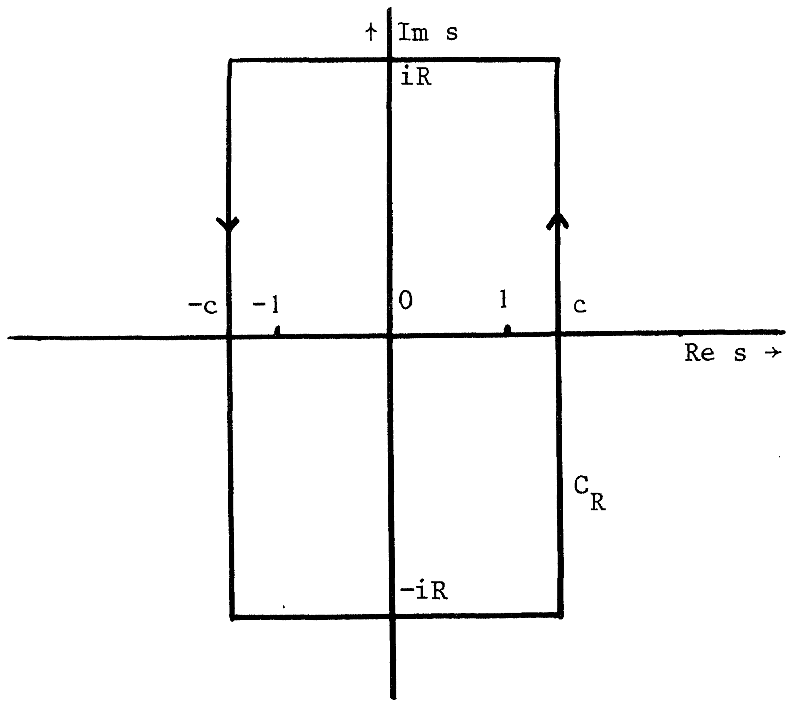
$$= [1 + s \log \frac{2\pi}{x} + O(s^2)][- \frac{1}{2s^2} + O(1)],$$

and the residue is $- \frac{1}{2} \log \frac{2\pi}{x}$.

After these preparations we consider the following integral

$$I_R(x) = \frac{1}{2\pi i} \int_{C_R} (2\pi/x)^s \, E(s)ds$$

where the contour C_R is a rectangle as drawn in the following picture:



The contributions from the horizontal parts of the contour vanish when R tends to ∞. This follows from the fact that the zeta functions in (4.2) are bounded by algebraic or logarithmic functions of Im s (see WHITTAKER & WATSON (1927, section 13.5, 13.51), whereas

$$\frac{1}{\cos(\frac{1}{2}\pi s)} = O(e^{-\frac{1}{2}\pi|\mathrm{Im}\ s|}), \qquad \mathrm{Im}\ s \to \pm\infty.$$

Hence,

$$\lim_{R\to\infty} I_R(x) = f(x) - g(x),$$

where f(x) is defined by (4.2) with c > 1 and g(x) by the same formula with c < -1. That is

$$(4.4) \qquad g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (2\pi/x)^s\ E(s)ds, \qquad c < -1.$$

On the other hand, we have, using the residues computed above,

$$I_R(x) = \frac{\pi^2}{6x} + \frac{1}{2}\log\frac{x}{2\pi} - \frac{x}{24}, \qquad R > 0.$$

Transforming in (4.4) s → -s and using again the symmetry of E(s) we obtain (with c > 1)

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (2\pi/x)^{-s}\ E(s)ds = f(4\pi^2/x),$$

which gives our main result

THEOREM. *The function* f(x) *defined in* (1.3) *satisfies the relation* (1.6), *where* x > 0.

REMARK. The function f can be defined for complex values at its argument. Then (1.3) is well defined when x is replaced by z = x + iy, with x > 0. It is not difficult to verify that also (1.6) and (3.9) hold true when x is replaced by z = x + iy, x > 0.

REFERENCES

OLVER, F.W.J., (1974), *Asymptotics and Special Functions*, Academic Press, New York.

SNEDDON, J.N., (1972), *The Use of Integral Transforms*, McGraw-Hill, New York.

WHITTAKER, E.T. & G.N. WATSON, (1927), *A course in modern Analysis*, Cambridge University Press.